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Stokes Parameters and Stokes Operators

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Abstract—It is shown that the description of polarization based on quantization of classical Stokes parameters is incomplete in quantum domain. For example, the polarization of the electric dipole radiation of the atomic or molecular transitions is described by nine generators of the SU(3) subalgebra in the Weyl–Heisenberg algebra of photons, in other words, by nine independent Stokes operators instead of four dependent classical Stokes parameters. Although some of the Stokes operators have one and the same averages, they describe absolutely different physical quantities with strongly different quantum fluctuations. The quantum polarization properties of the electric dipole and quadrupole radiation are examined as a function of distance for the near, intermediate, and far zones. Application of results in the near-field optics and quantum entanglement research is discussed.

1. INTRODUCTION

Light propagation lies at the heart of any optical phenomena, and has been analyzed since the early days of science of optics. Standard supposition in such analysis is to single out the generation of light by source and to look its spatial and temporal evolution free of source effects [1]. This evolution or propagation region is examined in three separate zones, namely near, intermediate and far zones of radiation, since light exhibits remarkably different behaviors in them [2]. A critical parameter in classifying the zones is the wavelength of the radiation. Only in the case of monochromatic light, a well-separated classification is possible. Another parameter is the dimension of the source, and for modern optical applications, typical light sources have dimensions much smaller than any distance of interest (atom, for example). This brings the concept of localized sources with additional assumption that it defines a closed system of charges and currents. Under these physical assumptions, light propagation from localized sources, in the framework of classical electrodynamics, is described in terms of multipole expansion. In this report, we follow the same ideology but from the point of view of quantum electrodynamics and particularly we study polarization properties of such multipole radiation.

The motivation of this study stems from several advances in modern communication and computation technologies [3] as well as developments of near-field optical devices, like optical scanning near-field microscopes (NSOM). As discrete degrees of freedom, polarization states of light considered to be a good basis for information coding. However, since polarization of light is a local property and information exchanges between quantum chips can occur in distances much smaller, or comparable to, wavelength, near field effects becomes important. In near field, light polarization also has a longitudinal component [4] which can bring a new key to logical basis of $|0_L\rangle$, $|1_L\rangle$. Furthermore, some of the information on the source parameters is trapped by the local, quasi-static mode around the source in the near zone and not all the information can reach to the far zone. Our studies will then also pose and answer the intriguing question how to extract all the information including those at the near zone by far zone measurements. Such a question is undoubtly very important also for analysis of NSOM data.

Organization of the report is as follows. In Section 2, conventional quantum optics in terms of plane wave photons will be reminded. In Section 3, radiation from a localized quantum source will be investigated and effective local polarization operators will be introduced. In Section 4, some quantities of quantum optics such as Mandel's Q-parameter, coherent states will be reinvestigated in terms of the effective polarization operators. In Section 5, a complete set of local Stokes operators which form a local SU(3) algebra will be introduced. In the last section, dipole field will be examined and connection between nine local Stokes operators and four Stokes parameters will be shown. The difference between plane wave photons and spherical photons, even in the far zone, will be shown.

2. CONVENTIONAL QUANTUM OPTICS

A photon with a fixed energy can be characterized by three quantum numbers [5]. Momentum $\hbar\mathbf{k}$ and a state of polarization, $\mathbf{e}_1$ and $\mathbf{e}_2$, orthogonal to each other and to the direction of propagation. The photon annihilation and creation operators for wavevector $\mathbf{k}$ and polarization $\mathbf{e}_\alpha$; $a_{\mathbf{k}\sigma}$, $a^+_{\mathbf{k}\sigma}$, satisfy the bosonic commutation relations:

$$[a_{k\sigma}, a_{k'\sigma'}] = 0, \quad [a_{k\sigma}, a^+_{k'\sigma'}] = \delta_{\sigma\sigma'}\delta_{kk'}.$$

This representation is relevant to a translationally invariant system which has no source.

3. RADIATION OF A LOCALIZED QUANTUM SOURCE

In order to consider the radiation from a localized quantum source, such as an atom, a molecule, which emits or absorbs photons, the momentum eigenbasis is not relevant since the translational symmetry of the system is broken. Let us replace the quantum source at the origin, then the system has a rotational symmetry. Since $[\mathbf{J}, H] = 0$, the most convenient quantum representation of the problem is provided by the photons with definite angular momentum and parity which are the so-called spherical photons [6]. In this case, a monochromatic photon with total angular momentum can be characterized by $2j + 1$ quantum numbers. In this representation, the photon operators are $a_{\lambda jm}(k)$, $a^{\dagger}_{\lambda jm}(k)$. Each photon in state $a_{\lambda jm}(k)$ has wavenumber k, angular momentum j, angular momentum in z-direction $m\hbar$ ($m = -j, \ldots, +j$) and parity $(-1)^{j+1}$ (for M-type) and $(-1)^j$ (for E-type). Since we will restrict our quantum source to a definite one, we will denote $a_{\lambda jm}(k) \equiv a_{jm}$. The bosonic commutation relations are satisfied for a_{jm}; $a^{\dagger}_{jm}$,

$$[a_{jm}, a^{\dagger}_{j'm'}] = \delta_{jj'}\delta_{mm'}.$$

The total angular momentum of a photon is $\mathbf{J} = \mathbf{S} + \mathbf{L}$, where $\mathbf{S}$ is spin operator and $\mathbf{L}$ is orbital angular momentum operator [7]. At a given $\mathbf{J}$ value, we can make the following expansion for positive frequency part of vector potential $\mathbf{A}$:

$$\mathbf{A}(\mathbf{r}) = \sum_{\mu=-1}^{1} \chi_{\mu} \sum_{m=-j}^{j} V_{\mu m}(\mathbf{r}) a_{jm}.$$

Here χ_{μ} are photon spin wavefunctions and satisfy orthonormality condition

$$\chi^{\dagger}_{\mu} \cdot \chi_{\mu'} = \delta_{\mu\mu'}.$$

$V_{\mu m}(\mathbf{r})$ are well-known functions composed of $f_j(kr)$ linear combinations of spherical Hankel functions, $Y_{lm}(\theta, \phi)$ spherical harmonics and Clebsch–Gordan coefficients [2, 6]. It can be both magnetic and electric type,

$$\begin{cases} V_{\mu m}(\mathbf{r}) = \sqrt{\dfrac{2\pi\hbar c}{kV}} \\ \times f_j(kr)\langle 1\, j\mu m - \mu | jm\rangle Y_{jm}(\theta, \phi)(\mathrm{M}) \\ \sqrt{\dfrac{j}{j+1}} f_{j+1}(kr)\langle 1\, j+1\, \mu m - \mu | jm\rangle Y_{j+1\, m-\mu}(\theta, \phi) \\ - f_{j-1}(kr)\langle 1\, j-1\, \mu m - \mu | jm\rangle Y_{j-1\, m-\mu}(\theta, \phi)(\mathrm{E}) \\ \int_0^R f_j(kr) f_j(k'r) r^2 dr = V\delta_{kk'}. \end{cases} \quad (1)$$

Any component of $\mathbf{A}(\mathbf{r})$ can be written as

$$A_{\mu}(\mathbf{r}) = \chi^{\dagger}_{\mu} \mathbf{A}(\mathbf{r}) = \sum_{m=-j}^{j} V_{\mu m}(\mathbf{r}) a_{jm}.$$

Let us now consider the commutator of $A_{\mu}(\mathbf{r})$ and $A^{\dagger}_{\mu'}(\mathbf{r})$ at the same space point. It is

$$[A_{\mu}(\mathbf{r}), A^{\dagger}_{\mu'}(\mathbf{r})] = \sum_{m=-j}^{j} V_{\mu m}(\mathbf{r}) V^{*}_{\mu' m}(\mathbf{r}) \equiv \mathcal{V}_{\mu\mu'}(\mathbf{r}),$$

where $\mathcal{V}_{\mu\mu'}(\mathbf{r})$ is a position-dependent (3×3) Hermitian matrix. There exists a local unitary transformation $U(\mathbf{r})$ for each $(\mathbf{r})$ which diagonalizes the matrix $\mathcal{V}(\mathbf{r})$ [8]:

$$\begin{cases} U(\mathbf{r})U^{\dagger}(\mathbf{r}) = 1 \\ U^{\dagger}(\mathbf{r})\mathcal{V}(\mathbf{r})U(\mathbf{r}) = W(\mathbf{r}) = \begin{bmatrix} W_{+} & 0 & 0 \\ 0 & W_0 & 0 \\ 0 & 0 & W_{-} \end{bmatrix}. \end{cases} \quad (2)$$

The diagonal elements of W matrix are real and positive. At this point, we can define an effective operator $a_{\mu}(\mathbf{r})$

$$\begin{aligned} a_{\mu}(\mathbf{r}) &= \frac{1}{\sqrt{W_{\mu}(\mathbf{r})}} \sum_{\mu'=-1}^{1} U^{*}_{\mu\mu'}(\mathbf{r}) A_{\mu'}(\mathbf{r}) \\ &= \frac{1}{\sqrt{W_{\mu}(\mathbf{r})}} \sum_{\mu'=-1}^{1} U^{*}_{\mu\mu'}(\mathbf{r}) \sum_{m=-j}^{j} V_{\mu' m}(\mathbf{r}) a_{jm}. \end{aligned} \quad (3)$$

The commutator of $a_{\mu}(\mathbf{r})$ and $a^{\dagger}_{\mu}(\mathbf{r})$ at the same space point yields bosonic commutation relations

$$\forall \mathbf{r} \quad [a_{\mu}(\mathbf{r}), a^{\dagger}_{\mu'}(\mathbf{r})] = \delta_{\mu\mu'}.$$

Since $a_{\mu}(\mathbf{r})$ is linear with respect to a_{jm}, all other commutators are zero. $a_{\mu}(\mathbf{r})$ and at $a^{\dagger}_{\mu}(\mathbf{r})$ can be interpreted as annihilation and creation operators of polarization of a field at a given point.

4. POSITION-DEPENDENT QUANTUM OPTICS

The coherent state, which is defined by the eigenvalue equation for the non-Hermitian annihilation operator, e.g., [5] with respect to the operators of photon with given projection of angular momentum,

$$|\alpha\rangle = \bigotimes_{m=-j}^{j} |\alpha_m\rangle; \qquad a_{jm}|\alpha_m\rangle = \alpha_m|\alpha_m\rangle$$

is also a coherent state with respect to $a_{\mu}(\mathbf{r})$ because of the linear relation between annihilation operators:

$$a_{\mu}(\mathbf{r})|\alpha\rangle = \alpha_{\mu}(\mathbf{r})|\alpha\rangle,$$

$$\alpha_\mu(\mathbf{r}) = \sum_{m=-j}^{j} \alpha_m V_{\mu m}(\mathbf{r}).$$

The position-dependent coherence parameter is of the following form:

$$\alpha_\mu(\mathbf{r}) = \frac{1}{\sqrt{W_\mu(\mathbf{r})}} \sum_{\mu'=-1}^{1} \sum_{m=-j}^{j} U^*_{\mu\mu'}(\mathbf{r}) V_{\mu' m}(\mathbf{r}) \alpha_{jm}. \quad (4)$$

By using the local operators, the notions of quantum optics can be reinvestigated. For example, the normalized variance of the photon distribution, i.e., Mandel's Q-parameter which gives the statistics of photons is a position dependent parameter now [5]. Hence a local Mandel's Q-parameter should be redefined as

$$Q_\mu(\mathbf{r}) = \frac{\langle [\Delta a^\dagger_\mu(\mathbf{r}) a_\mu(\mathbf{r})]^2 \rangle - \langle a^\dagger_\mu(\mathbf{r}) a_\mu(\mathbf{r}) \rangle}{\langle a^\dagger_\mu(\mathbf{r}) a_\mu(\mathbf{r}) \rangle}. \quad (5)$$

Even though the parameter is a local one, it gives a global property of the field, namely, its statistics. For example, for the coherent state, $Q_\mu(\mathbf{r}) = 0$ everywhere which implies a Poissonian distribution for coherent state and a global property of the radiation field.

5. LOCAL STOKES OPERATORS

Let us consider now the polarization properties of the quantum multipole radiation for an arbitrary j, which is not necessarily one (dipole case). Polarization, the spatial anisotropy of the field, is a *local* property of the field. The polarization matrix is a local 3×3 Hermitian matrix. One can reconstruct the local polarization matrix in terms of effective creation and annihilation operators of polarization

$$P(\mathbf{r}) = \left\| A^\dagger_\mu(\mathbf{r}) A_{\mu'}(\mathbf{r}) \right\| \longrightarrow \left\| a^\dagger_\mu(\mathbf{r}) a_{\mu'}(\mathbf{r}) \right\|. \quad (6)$$

Note that the operators at $a^\dagger_\mu(\mathbf{r})$, $a_\mu(\mathbf{r})$ are quite different from the $a^\dagger_{jm}$, a_{jm} operators. The operators a_{jm} describing the multipole photons are independent of position, they act in $2j + 1$ dimensional space which coincide with three dimensional space only in the case of dipole photons. $a_{\mu'}(\mathbf{r})$ are local operators acting in three dimensional space and take into account the spatial properties as well as the quantum nature of multipole radiation at any distance from the source. These operators coincide with the dipole photons in the generation zone.

These operators can be used to construct the near and intermediate field zone quantum optics in addition to far zone quantum optics. One can write the local Stokes operators in this way, from the generators of the local SU(3) algebra in the Weyl–Heisenberg algebra of photons, describing the independent Hermitian bilinear forms in the creation and annihilation operators. The local Stokes operators are the following:

$$\begin{cases} S_0(\mathbf{r}) = \sum_\mu a^\dagger_\mu(\mathbf{r}) a_\mu(\mathbf{r}) \\ S_1(\mathbf{r}) = \mathscr{E}(\mathbf{r}) + \mathscr{E}^\dagger(\mathbf{r}) \\ S_2(\mathbf{r}) = -i[\mathscr{E}(\mathbf{r}) - \mathscr{E}^\dagger(\mathbf{r})] \\ S_3(\mathbf{r}) = a^\dagger_+(\mathbf{r}) a_+(\mathbf{r}) - a^\dagger_-(\mathbf{r}) a_-(\mathbf{r}) \\ S_4(\mathbf{r}) = a^\dagger_+(\mathbf{r}) a_+(\mathbf{r}) + a^\dagger_-(\mathbf{r}) a_-(\mathbf{r}) - 2a^\dagger_0(\mathbf{r}) a_0(\mathbf{r}) \\ S_5(\mathbf{r}) = a^\dagger_+(\mathbf{r}) a_0(\mathbf{r}) + \text{H.c.} \\ S_6(\mathbf{r}) = -i[a^\dagger_+(\mathbf{r}) a_0(\mathbf{r}) - \text{H.c.}] \\ S_7(\mathbf{r}) = a^\dagger_0(\mathbf{r}) a_-(\mathbf{r}) + \text{H.c.} \\ S_8(\mathbf{r}) = -i[a^\dagger_0(\mathbf{r}) a_-(\mathbf{r}) - \text{H.c.}] \\ \mathscr{E}(\mathbf{r}) = a^\dagger_+(\mathbf{r}) a_0(\mathbf{r}) + a^\dagger_0(\mathbf{r}) a_-(\mathbf{r}) + a^\dagger_-(\mathbf{r}) a_+(\mathbf{r}). \end{cases} \quad (7)$$

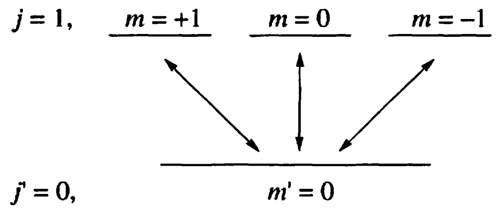
A careful investigation showed that there are nine Stokes operators. The expectation values of these operators over physical states will give the following information. $S_0(\mathbf{r})$ is the local intensity of the field. $S_1(\mathbf{r})$ and $S_2(\mathbf{r})$ are the claimants of phase information, relative phase angles, Δ_{+-}, Δ_{+0}, Δ_{0-}, where

$$\Delta_{\alpha\beta} = \arg A_\alpha(\mathbf{r}) - \arg A_\beta(\mathbf{r}). \quad (8)$$

$S_3(\mathbf{r})$ gives the local preponderance of positive helicity over negative helicity and $S_4(\mathbf{r})$ the preponderance of circular polarization over longitudinal polarization. $S_5(\mathbf{r})$, $S_6(\mathbf{r})$ and $S_7(\mathbf{r})$, $S_8(\mathbf{r})$ gives phase information about Δ_{+0} and Δ_{-0} respectively. The commutator $[S_1(\mathbf{r}), S_2(\mathbf{r})] = 0$, so that the corresponding physical quantities can be measured at any point at once.

6. DIPOLE FIELD POLARIZATION IN LOCAL PICTURE

Let us assume that a dipole atom is located at the origin of the coordinate system [9]. In the generation zone, one has $\mu = m$. In the near and intermediate zone, $\mu = -1, 0, 1$. But in the far zone, since $V_{\mu=0,m}$ vanishes, the intensity of longitudinally polarized component of the dipole radiation tends to zero. That means in the far zone, the radial component $\mu = 0$ is in the vacuum state. In this case, the expectation values of generalized



A dipole atom.

Stokes operators are the following:

$$\begin{cases} \langle S_0(\mathbf{r})\rangle = \sum\limits_{\mu=\pm} \langle a_\mu^\dagger(\mathbf{r}) a_\mu(\mathbf{r})\rangle \\ \langle S_1(\mathbf{r})\rangle = 2\,\mathrm{Re}\langle a_-^\dagger(\mathbf{r}) a_+(\mathbf{r})\rangle \\ \langle S_2(\mathbf{r})\rangle = 2\,\mathrm{Im}\langle a_-^\dagger(\mathbf{r}) a_+(\mathbf{r})\rangle \\ \langle S_3(\mathbf{r})\rangle = \langle a_+^\dagger(\mathbf{r}) a_+(\mathbf{r})\rangle - \langle a_-^\dagger(\mathbf{r}) a_-(\mathbf{r})\rangle \\ \langle S_4(\mathbf{r})\rangle = \langle S_0(\mathbf{r})\rangle \\ \langle S_5(\mathbf{r})\rangle = \langle S_6(\mathbf{r})\rangle = \langle S_7(\mathbf{r})\rangle = \langle S_8(\mathbf{r})\rangle = 0 \end{cases} \tag{9}$$

which coincide with classical Stokes parameters determined in the circular polarization basis. Hence, the polarization of quantum dipole radiation at far zone looks like that of the plane wave photons.

But there is a very fundamental difference in the quantum fluctuations of generalized Stokes operators in the far zone and conventional Stokes operators. Let us consider the variance of $S_1(\mathbf{r})$. The fluctuation for conventional Stokes operator S_1 is

$$\begin{aligned} \langle(\Delta S_1)^2\rangle &= 2\,\mathrm{Re}\langle(\Delta a_-^\dagger a_+)^2\rangle \\ &+ 2(\langle n_+ n_- - |\langle a_+^\dagger a_-\rangle|^2) + \langle n_+\rangle + \langle n_-\rangle \end{aligned} \tag{10}$$

and for generalized Stokes operator, but radial mode in vacuum, i.e., in the far zone

$$\begin{aligned} \langle(\Delta S_1)^2\rangle &= 2\,\mathrm{Re}\langle(\Delta a_-^\dagger a_+)^2\rangle \\ &+ 2\Big(\langle n_+ n_- - |\langle a_+^\dagger a_-\rangle|^2\Big) + \langle n_+\rangle + \langle n_-\rangle \\ &+ 2\,\mathrm{Re}\langle a_+^\dagger a_-\rangle + \langle n_+\rangle + \langle n_-\rangle. \end{aligned} \tag{11}$$

The underlined term arises even though the radial polarization is in vacuum, but nevertheless it exist and the additional three terms come from the commutation relations. The presence of these terms increases the quantum fluctuations of transversal polarization and changes them qualitatively since the term includes $2\,\mathrm{Re}\langle a_+^\dagger a_-\rangle$, a phase dependence.

This result shows us that the use of plane waves of photons rather than the spherical waves of photon can lead to a wrong result even in the far zone. It is worse to use the plane waves of photons in the near and intermediate zone, where the radial component of the field is no more in vacuum. Let us also note that the quantum fluctuations of polarization are very important in the quantum entanglement research since the existence of radial field, even in the vacuum state, increases the noise in the system.

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